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LETTER TO THE EDITOR

A statistical model for simulating miscible viscous fingers in porous media and other growth phenomena

Habib Sideiqui† and Muhammad Sahimi

Department of Chemical Engineering, University of Southern California, Los Angeles, CA 90089-1211, USA

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Abstract. A general stochastic model of cluster growth is proposed. In two special limits the model is reduced to the diffusion-limited aggregation (DLA) and the Eden models. By varying a physical parameter, the model interpolates smoothly between the DLA and the Eden model, and is also capable of simulating miscible viscous fingers in a porous medium, when dispersion effects are absent, in which the ratio of the viscosities of the displaced and displacing fluids is finite. The predictions agree with previous results obtained by a deterministic model. The model also has close connections with a variety of other models of growth phenomena.

Various non-equilibrium growth models have been studied actively during the last few years [1, 2] because of their relevance to a wide variety of phenomena of scientific and industrial interest, such as the growth of crystals from an undercooled melt or a supersaturated solution, coagulation of smoke particles, growth of tumours, turbulence, dielectric breakdown of composite solids and displacement of one fluid by another. The simplest of such growth models are perhaps the Eden model [3] and the diffusion-limited aggregation (DLA) model of Witten and Sander [4]. In the Eden model one starts from an occupied site of a large lattice, and at every step occupies one other site, which is selected randomly from the set of sites, called perimeter sites, which have at least one nearest-neighbour occupied site. This simple process allows every unoccupied site again and again the chance to become occupied. Therefore, after a long time, every site of the lattice will finally become occupied. Thus, if one characterises this model with a fractal dimension D, then $D = d$, where d is the Euclidean dimension of the lattice. In a variant of this model [5], if a perimeter site of the lattice is picked at random but does not become part of the growing cluster (i.e., the probability of its joining the cluster is less than one), it remains unoccupied forever. The clusters grown in this way no longer have trivial structures and, in fact, have close connections with a variety of complex dynamical processes such as directed percolation and cellular automata.

In the DLA model the initial state is a seed particle located at the centre of a large lattice. A second particle is released at the surface of this lattice, which performs a random walk until it reaches a site adjacent to the seed site, where it stops its random walk and joins the cluster. The process of releasing the particles is continued until a large cluster is formed. This model has been extensively studied in the past few years

† Present address: Rohm and Haas Co, Bristol Research Laboratory, Building 147C, Bristol, PA 19007, USA.

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[2]. It is now generally believed that the DLA model has a fractal dimension of $D \simeq 1.7$ in 2D.

Paterson [6] pointed out that the DLA model is essentially equivalent to the problem of a miscible displacement in a porous medium in which a fluid of viscosity μ_1 displaces another fluid of viscosity μ_2, with $m = \mu_2/\mu_1 \to \infty$. This process is always unstable and is characterised by the formation of a multitude of viscous fingers (VFs) of the displacing fluid penentrating through the displaced fluid. The analogy between the DLA model and VFs is appropriate when $m \to \infty$, and dispersion (mixing) effects are completely absent. Experimental works [7-9] have confirmed the qualitative similarities between the DLA model and VFs in the limit $m \to \infty$. Theoretical studies [10-12] have indicated that although the DLA model and VFs may be characterised by approximately the same fractal dimensionality D, the sweep efficiencies of the two processes, namely, the volume fraction of the pore space swept by the displacing fluid, may not be equal, so that the analogy between the two phenomena is not generally exact. Despite this, the DLA model can often provide a good approximation to the problem of VFs in a porous medium.

While there have been several studies of VFs for finite values of m [13-17], there has not been any rigorous attempt to generalise the DLA model in order to simulate VFs, for finite values of m, using *only* random walk methods. Instead, these studies [13-16] have relied on the explicit calculations of the flow field, and the advancement of the displacing fluid based on a stochastic rule, which is essentially equivalent to a model of dielectric breakdown of composite solids [18], or a deterministic rule [17]. An earlier attempt by Sahimi and Yortsos [19] to simulate VFs, for finite values of m, by random walk methods is not generally exact. Kadanoff [20] has developed a rigorous method for simultating VFs in a *Hele-Shaw cell.* However, his method is only appropriate for the Hele-Shaw cell in *the limit* $m \to \infty$, and when *surface tension is present* (i.e., an *immiscible* displacement). From a computational point of view, it would be highly desirable to develop a random walk algorithm for studying VFs for finite values of m, since such an algorithm can be very efficient in terms of computer time and memory. The purpose of this letter is to develop such an algorithm. We also show that this algorithm is a general model of cluster growth for which the Eden and DLA models represent two special limits. As such the model is the first of its kind.

We first note that, without dispersion and surface tension effects, it suffices to find the pressure field in the regions of displacing and displaced fluids, which obeys

$$\nabla(k\nabla P_i) = 0 \qquad i = 1, 2 \tag{1}$$

where k is a position-dependent local permeability, P the pressure, and subscripts 1 and 2 refer to the displacing and displaced fluid, respectively. The boundary between the two fluids advances according to Darcy's law

$$\boldsymbol{u} = -\frac{k}{\mu}\nabla \boldsymbol{P} \tag{2}$$

where $\boldsymbol{u}$ is the fluid flux. Across the front that separates the two fluids, pressure and fluid fluxes must be continuous, $P_1 = P_2$ and $u_1 = u_2$, where these quantities are measured just inside regions of fluids 1 and 2.

Consider two points a and b on two sides of the front, but very close to it. If we discretise the pressure field on both sides of the front with equal spacing Δx, we obtain $\partial P_1/\partial x \simeq (P_f - P_a)/\Delta x$, and $\partial P_2/\partial x \simeq (P_b - P_f)/\Delta x$, where P_a, P_f, and P_b are the pressures at a, on the front and at b, respectively. Using the fact that the fluid fluxes must

be continuous across the front, we obtain $M(P_a - P_f) = P_f - P_b$, from which the pressure on the front is found, $P_f = (MP_a + P_b)/(M+1)$, where $M = (k_1\mu_2)/(k_2\mu_1)$ is the mobility ratio. Similar to the DLA model, we can introduce the probability p that the front advances one pore. According to equation (2), p is proportional to the fluid flux in a pore adjacent to the front. Because the effective permeabilities k_1 and k_2 experienced by the fluids on two sides of the front are essentially equal (since the medium is macroscopically homogeneous), we have $M \simeq m$. However, because of the viscosity contrast between the two fluids, the pressure distributions in regions 1 and 2 are not the same. Thus, the probability of advancing the front should depend on the local pressures at points *a* and *b*. Hence, we generalise the DLA model and introduce *two* probabilities p_1 and p_2 for the advancement of the front, if one approaches it from regions 1 and 2, respectively. Using equation (2) and the expression for p_f, we obtain $p_2/p_1 = M$, and because $0 \leq p_1 \leq 1$ and $0 \leq p_2 \leq 1$, we finally obtain

$$p_1 = (M+1)^{-1} \tag{3}$$

and

$$p_2 = M(M+1)^{-1}. \tag{4}$$

In the language of the DLA model, p_1 and p_2 are the *sticking* probabilities, i.e., the probabilities that the random walkers stick to the growing aggregate, once they are adjacent to the front, and advance it by one pore. Therefore, a random walk algorithm to simualte VFs for finite values of mobility ratio M may be as follows. We inject *two* types of random walkers into the system. One of them is injected in the region of fluid 2, at the outer boundary of the system, whereas the other one is injected in the region of fluid 1 at the source where fluid 1 is injected into the medium, and each particle executes its random walk in its own fluid region (so that equation (1) is satisfied in each region). The motion of the particle in the region of fluid 2 is similar to the DLA model [2] (except that its sticking probability is less than one). The addition of another type of random walker to the region of fluid 1 is the new feature of the model, absent from all prevous models, which is necessary if m is finite. In the beginning of the simulation, the motion of the particle in the region of fluid 1 is affected by the small size of the cluster of fluid 1. However, as the displacement front advances this effect vanishes, and the long time behaviour of the system remains unaffected by this.

If a particle attempts to cross the front from the region of fluid 2, then the last site in that region that was visited by the random walker and the bond that connects it to the front are filled with the displacing fluid with probability p_2, and the particle is removed from the system. However, if the particle does not join the front, it continues its journey. Similarly, if a particle attempts to cross the front from the region of fluid 1, then, the first visited site in the region of fluid 2 and the bond that connects it to the front are invaded by the displacing fluid with probability p_2, and the particle is removed from the system. But if this move is rejected, the particle continues its motion. Thus, according to this algorithm, the front always advances forward, which is physically the case as long as surface tension and dispersion are absent, which is the case here. Note that in the limit $M \to \infty$, the algorithm is exactly equivalent to the DLA model (since $p_1 \to 0$). Moroever, in the opposite limit $M \to 0$, the pressure in the region of displaced fluid is constant and one needs only one type of random walker (moving on the cluster of displacing fluid) to simulate the process. Note also that since $p_1/p_2 = M = n_1/n_2$, where n_1 and n_2 are integers, for every n_1 particles that are released in the region of fluid 2, we must release n_2 particles in the region of fluid 1 (or,

equivalently, for every n_1 steps that a particle takes in the region of fluid 2, the particle in the region of fluid 1 must take n_2 steps). This reflects the fact that the pressure distributions in the two fluid regions and the mobility (k/μ) of the fluids are not equal.

The limit $p_1 \to 1$ represents a random walk version of the Eden model: each particle starts its motion from the centre of the cluster, executes its random walk *on the cluster*, and once it reaches an unoccupied site it adds it to the cluster. In one and two dimensions, the probability that a random walker, starting at the origin, will visit every site of a lattice is one [21] and, therefore, similar to the Eden model every unoccupied site will eventually become occupied and part of the growing cluster. In three or higher dimensions, the probability that every site is visited by an unrestricted random walker is less than one. However, in the present problem, once a random walker visits an unoccupied site and adds it to the growing cluster, the random walker is removed and a new walk at the origin of the cluster is initiated. It can be shown that in such a process, every site will eventually be visited by at least one random walker and, therefore, in the limit of long times, the growing cluster will be completely compact as in the Eden model. Our computer simulations support this.

We should mention that if the porous medium is characterised by a pore size of pore conductance distribution, then the particles take each step of their random walks with a probability porportional to the pore conductance. As such, the model represents a growth phenomenon in a *disordered* porous medium. We also note that Leclerc and Neale [22] have also proposed a two-particle model for simulating flow in porous media, although the origin of their laws of the motion of the particles and their relation with ours is not clear to us.

We have used a triangular lattice to simulate this model for various values of M. Each bond of the lattice represents a pore, and we assume that all pores have the same effective radius. As the experiments of Chen and Wilkinson [7] demonstrated, in an ordered medium and in the absence of dispersion, for large values of M, miscible displacement of one fluid by another results in completely ordered (dendritic) patterns, whereas the model we have described (and the DLA model) predicts random and very branched patterns. The reason is that [23] such models are dominated by the noise that is generated by the random trajectories of the particles. Only the most exposed part of the growing aggregate absorbs the incoming particles. To eliminate this noise, one uses a method due to Tang [24]: one counts the number S_i of random walkers vising the perimeter site i. If S_i reaches a preassigned value S, i becomes part of the aggregate (of the displacing fluid). Once i belongs to the aggregate, S_j is *not* set to be zero for all other sites j and the counting process continues. The limit $S = 1$ corresponds to the DLA model, whereas in the limit $S \to \infty$, there would be no noise in the system (i.e., the mean-field limit). We used this method of noise reduction with our model.

Figure 1 shows the patterns of the displacing fluid for $M = 5$ and 1000 and $S = 100$. They are in perfect agreement with those obtained by King [17] using a fully deterministic model (based on explicit calculation of the flow field) with no noise. This agreement confirms the validity of our model. As can be seen, in the absence of noise the anisotropy of the lattice totally dominates the patterns. For $M < 1$, the displacement process is stable, and the resulting aggregates are completely compact with a fractal dimension $D = 2$. For $M \geqslant 1$, the interior of the cluster of the displacing fluid is compact for short-length scales, and one has a compact cluster with $D = 2$, independent of the value of M (this means that if we consider only the interior of the cluster of the displacing fluid, the sweep efficiency is 100%). This is because if for short-length

Figure 1. Patterns of viscous fingers for mobility ratio $M = 5$ (top) and $M = 1000$ (bottom), obtained on a triangular network.

scales $D < 2$, then the density of the displacing fluid would vanish as the displacement proceeds. This means that very thin sections of the displacing fluid would have to support a vast tenuous network of fingers at the tips. In fact, if the displaced fluid has a finite mobility, one would expect fingers to become thicker. However, for $M > 1$ the displacement is unstable, if dispersion effects are absent, and we cannot expect the growing cluster to be compact, since there is no intermediate length scale between the size of the system and that of a pore. Thus, as pointed out by King [17], only the surface of the cluster of the displacing fluid can have a fractal-like character.

To calculate the fractal dimension D_s of the surface sites, i.e., those sites that have one or more nearest-neighbour sites occupied by the displaced fluid and which belong to the front, we used the two-point correlation function $\rho(r)$

$$\rho(r) \sim \frac{1}{r} \sum_i N_i(r) \tag{5}$$

where $N_i(r)$ is the number of surface sites in a shell around site i between r and $r + \Delta r$. One must have $\rho(r) \sim r^{D_s - d}$. The results for $\rho(r)$ are shown in figure 2, while the resulting D_s, for various values of M, are shown in figure 3, where they are compared with those of King [17] who used a deterministic model with no noise. The agreement is very good, confirming again the validity of our model. In a future paper [25], we shall show that the predictions of this model are in agreement with the experimental measurements of the sweep efficiency of miscible displacements in model porous media.

As mentioned above, the limit $M \to 0$ ($p_2 \to 0$) represents a random walk version of the Eden model. The clusters that we obtain in this limit are all round and compact, as expected. A quantity of interest in the Eden model is [26] the thickness W of its surface defined through

$$W^2 = \langle r^2 \rangle - \langle r \rangle^2 \tag{6}$$

where r is the distance between the unoccupied perimeter sites next to the front and the origin of the cluster. This quantity has been shown (see Zabolitzky and Stauffer [27] and references therein) to have a peculiar behaviour, even though the Eden clusters seem to have a trivial structure. Theoretically, one expects that at $d = 2$, $W \sim N_s^{1/2}$, where N_s is the number of sites in the cluster. However, even simulations with very large clusters [26] have shown some deviations from this prediction, and the behaviour

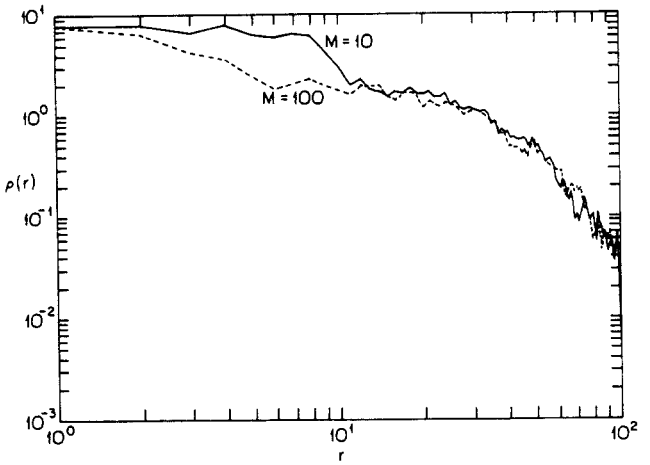

Figure 2. Two-point correlation function $\rho(r)$ as a function of radius r for various values of M.

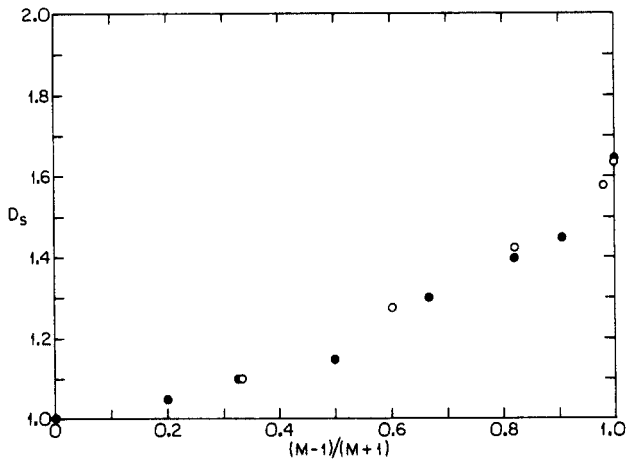

Figure 3. Surface fractal dimension D_s as a function of M. Open circles are the results of our simulations, while full circles are those of King [17].

of W for $d > 2$ is particularly anomalous. We have found that if one uses our random walk version of the Eden model, the scaling law $W \sim N_s^{1/2}$ may be reached relatively fast even with relatively small clusters. Thus, this random walk model might prove a better means of studying the peculiar behaviour of the surface of the Eden clusters. In a future paper, we shall report the results of such a study.

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